ON EXCEPTIONAL EIGENVALUES OF THE LAPLACIAN FOR $\Gamma_0(N)$

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Abstract. An explicit Dirichlet series is obtained, which represents an analytic function of s in the half-plane $\Re s > 1/2$ except for having simple poles at points s_j that correspond to exceptional eigenvalues λ_j of the non-Euclidean Laplacian for Hecke congruence subgroups $\Gamma_0(N)$ by the relation $\lambda_j = s_j(1-s_j)$ for $j = 1, 2, \cdots, S$. Coefficients of the Dirichlet series involve all class numbers h_d of real quadratic number fields. But, only the terms with $h_d \gg d^{1/2-\epsilon}$ for sufficiently large discriminants d contribute to the residues $m_j/2$ of the Dirichlet series at the poles s_j, where m_j is the multiplicity of the eigenvalue λ_j for $j = 1, 2, \cdots, S$. This may indicate (I'm not able to prove yet) that the multiplicity of exceptional eigenvalues can be arbitrarily large. On the other hand, by density theorem [3] the multiplicity of exceptional eigenvalues is bounded above by a constant depending only on N.

1. Introduction

Let N be a positive integer. Denote by $\Gamma_0(N)$ the Hecke congruence subgroup of level N. The non-Euclidean Laplacian Δ on the upper half-plane $\mathcal{H}$ is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Let D be the fundamental domain of $\Gamma_0(N)$. Eigenfunctions of the discrete spectrum of Δ are nonzero real-analytic solutions of the equation $\Delta\psi = \lambda\psi$ such that $\psi(\gamma z) = \psi(z)$ for all γ in $\Gamma_0(N)$ and such that ψ is square integrable on D with respect to the Poincaré measure dz of the upper half-plane.

The Hecke operators T_n, $n = 1, 2, \cdots$, $(n, N) = 1$, which act in the space of automorphic functions with respect to $\Gamma_0(N)$, are defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n,\, 0\le b<d} f\left(\frac{az+b}{d}\right).$$

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It is well-known (see Iwaniec [3]) that there exists a maximal orthonormal system of eigenfunctions of Δ such that each of them is an eigenfunction of all the Hecke operators. Let λ_j, $j = 1, 2, \cdots$, be an enumeration in increasing order of all positive discrete eigenvalues of Δ for $\Gamma_0(N)$ with an eigenvalue of multiplicity m appearing m times, and let $\kappa_j = \sqrt{\lambda_j - 1/4}$ with $\Im\kappa_j > 0$ if $\lambda_j < 1/4$.

If λ is a positive discrete eigenvalue less than $1/4$, we call it an exceptional eigenvalue. Let $\lambda_1, \cdots, \lambda_S$ be exceptional eigenvalues of the Laplacian Δ for $\Gamma_0(N)$.

In 1965, A. Selberg [10] made the following fundamental conjecture.

Selberg's eigenvalue conjecture. *If λ is a nonzero discrete eigenvalue of the non-Euclidean Laplacian for any congruence subgroup, then $\lambda \geq 1/4$.*

A. Selberg [10] proved that $\lambda \geqslant 3/16$. The best available lower bound $\lambda \geq 975/4096$ is due to Kim and Sarnak [4]. It was obtained by combining automorphic lifts $\mathrm{sym}^3 : GL(2) \to GL(4)$ [5] and $\mathrm{sym}^4 : GL(2) \to GL(5)$ [4] with families of L-functions [8]. We note that if the general functorial conjectures concerning the automorphic lifts $\mathrm{sym}^k : GL(2) \to GL(k+1)$ are true for all $k > 1$, then Selberg's eigenvalue conjecture would follow.

In this paper, we indicate an elementary approach towards the Selberg eigenvalue conjecture. Namely, we prove the following theorem.

Theorem 1. *Let*

$$L(s) = \sum_{k|N} \sum_{d\in\Omega,\, k|u_d} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1 + (\frac{d}{p}))\frac{h_{dk^2}\ln\epsilon_{dk^2}}{(du_d^2)^s} - \sum_{d\in\Omega}\frac{h_d\ln\epsilon_d}{(du_d^2)^s}$$

where (v_d, u_d) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$ and the product on p^{2l} is over all distinct primes p with p^{2l} being the greatest even p-power factor of $(d, N/k)$. Then $L(s)$ represents an analytic function of s in the half-plane $\Re s > 1/2$ except for having simple poles at $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$. Moreover, we have

$$m_j = 2\, Res_{s=s_j} L(s)$$

for $j = 1, 2, \cdots, S$.

Corollary 2. *If N is square free, then the series*

$$L_1(s) = \sum_{\substack{m|N,\, k|N \\ (m,k)\neq(1,1)}} k\frac{\mu((m,k))}{(m,k)} \sum_{d\in\Omega,\, k|u_d} \left(\frac{d}{m}\right)\frac{h_d\ln\epsilon_d}{(du_d^2)^s}$$

represents an analytic function of s in the half-plane $\Re s > 1/2$ except for having simple poles at $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$, where (v_d, u_d) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$. Moreover, we have

$$m_j = 2\, Res_{s=s_j} L_1(s).$$

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2. Proofs of Theorem 1 and Corollary 2

We denote by h_d the class number of indefinite rational quadratic forms with discriminant d. Let

$$\epsilon_d = \frac{v_0 + u_0\sqrt{d}}{2},$$

where the pair (v_0, u_0) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$. Let Ω be the set of all the positive integers d such that $d \equiv 0$ or $1 \pmod 4$ and such that d is not a square of an integer.

Lemma 2.1. *Let d and d_1 be integers in Ω. If $d_1 = dl^2$, then*

$$h_{d_1} \ln \epsilon_{d_1} = l \prod_{p|l} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right) h_d \ln \epsilon_d.$$

Proof. The stated identity follows from Dirichlet's class number formula (see, §100 of Dirichlet [2])

$$h_{d_1} \ln \epsilon_{d_1} = \sqrt{d_1} L(1, \chi_{d_1})$$

and the identity

$$L(1, \chi_{d_1}) = L(1, \chi_d) \prod_{p|l} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right). \quad \square$$

Lemma 2.2. *Let d and d_1 be integers in Ω, and let $d_1 = dl^2$. Then $\epsilon_{d_1} = \epsilon_d^{\nu_l}$ for a positive integer ν_l.*

Proof. If (v_1, u_1) is the smallest positive solution of Pell's equation

$$v^2 - dl^2u^2 = 4, \tag{2.1}$$

then

$$\epsilon_{d_1} = \frac{v_1 + \sqrt{d_1}u_1}{2}.$$

Let (v_0, u_0) be the smallest positive solution of Pell's equation

$$v^2 - du^2 = 4. \tag{2.2}$$

By §85 of Dirichlet [2], all positive solutions (v, u) of (2.2) are given by the formula

$$\frac{v + \sqrt{d}u}{2} = \left(\frac{v_0 + \sqrt{d}u_0}{2}\right)^n$$

for positive integers n. Since (v_1, lu_1) is a positive solution of (2.2), there exists a positive integer ν_l such that

$$\frac{v_1 + \sqrt{d_1}u_1}{2} = \left(\frac{v_0 + \sqrt{d}u_0}{2}\right)^{\nu_l}. \quad \square$$

We denote the multiplicity of the eigenvalue λ_j by m_j for $j = 1, 2, \cdots$.

Lemma 2.3. *Let N be any positive integer, and let*

$$L_N(s) = \sum_{k|N} k^{1-2s} \sum_{d\in\Omega} \sum_{u} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1+(\frac{d}{p})) \prod_{p|k} (1-(\frac{d}{p})p^{-1}) \frac{h_d \ln \epsilon_d}{(du^2)^s}$$

for $\Re s > 1$, where the sum on u is over all positive integers u such that $\sqrt{4+dk^2u^2} \in \mathbb{Z}$ and where the product on p^{2l} is over all distinct primes p with p^{2l} being the greatest even p-power factor of $(d, N/k)$. Then $L_N(s)$ is analytic for $\Re s > 1$ and has analytic continuation to the half-plane $\Re s > 1/2$ except for having simple poles at $s = 1$ and $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$. Moreover, we have

$$m_j = 2Res_{s=s_j} L_N(s)$$

for $j = 1, 2, \cdots, S$.

Proof. Let

$$h(r) = 4^s \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \int_0^\infty \left(u + \frac{1}{u} + 2\right)^{1/2-s} u^{ir-1} du$$

for $\Re s > 1/2$. Then the lemma follows from Theorem 4.3, the proof of Lemma 5.3, the proof of Theorem 1 in Li [7], and the Selberg trace formula

$$\begin{aligned} h(-i/2) + \sum_{j=1}^\infty h(\kappa_j) m_j \\ = 4^{1/2+s} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} L_N(s) + f(s) \end{aligned}$$

for $\Re s > 1$ where $f(s)$ is a certain analytic function of s in the half-plane $\Re s \geq 1/2$ except for a possible pole at $s = 1/2$ (see (4.4) of Li [7]). $\square$

Remark 2.4. Siegel [11] proved that

$$\lim_{d\to\infty} \frac{\ln(h_d \ln \epsilon_d)}{\ln d} = \frac{1}{2}. \tag{2.3}$$

Lemma 2.5. *Let*

$$\bar{L}_N(s) = \sum_{k|N} \sum_{d\in\Omega,\, k|u_d} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1+(\frac{d}{p})) \frac{h_{dk^2} \ln \epsilon_{dk^2}}{(du_d^2)^s}$$

for $\Re s > 1$, where (v_d, u_d) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$ and the product on p^{2l} is over all distinct primes p with p^{2l} being the greatest even p-power factor of $(d, N/k)$. Then $\bar{L}_N(s)$ is analytic for $\Re s > 1$ and

has analytic continuation to the half-plane $\Re s > 1/2$ except for having simple poles at $s = 1$ and $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$. Moreover, we have

$$m_j = 2Res_{s=s_j}\bar{L}_N(s).$$

Proof. By Lemma 2.3, the function

$$L_N(s) = \sum_{k|N} k^{1-2s} \sum_{d\in\Omega}\sum_{u} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k}(1+(\frac{d}{p}))\prod_{p|k}(1-(\frac{d}{p})p^{-1})\frac{h_d \ln \epsilon_d}{(du^2)^s} \tag{2.4}$$

has analytic continuation to the half-plane $\Re s > 1/2$ except for having simple poles at $s = 1$ and $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$, where the sum on u is over all positive solutions of Pell's equation

$$v^2 - dk^2u^2 = 4. \tag{2.5}$$

Let (v_k, u_k) be the smallest positive solution of (2.5). By §85 of Dirichlet [2], all positive solutions (v, u) of (2.5) are given by the formula

$$\frac{v + \sqrt{d}ku}{2} = \left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^n$$

for $n = 1, 2, \cdots$. Hence, we have

$$\begin{aligned}\sqrt{d}ku &= \left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^n \left(1 - \left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^{-2n}\right)\\ &> \left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^n (1 + 2/\sqrt{d}k)^{-1}.\end{aligned} \tag{2.6}$$

Let $\sigma = \Re s > 1/2$, and let $\tau(n)$ be the number of positive divisors of an integer n. By (2.6) and (2.3), we have

$$\begin{aligned}&|\sum_{k|N} k^{1-2s} \sum_{d\in\Omega}\sum_{u\neq u_k} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k}(1+(\frac{d}{p}))\prod_{p|k}(1-(\frac{d}{p})p^{-1})\frac{h_d \ln \epsilon_d}{(du^2)^s}|\\ &\le \sum_{k|N}\sqrt{kN}2^{\tau(N)}\sum_{d\in\Omega}(1+2/\sqrt{d}k)^{2\sigma}h_d\ln\epsilon_d\sum_{n=2}^{\infty}\left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^{-2n\sigma}\\ &\le \sum_{k|N}\sqrt{kN}2^{\tau(N)}3^{2\sigma+1}\sum_{d\in\Omega}h_d\ln\epsilon_d\left(\frac{v_k + \sqrt{d}ku_k}{2}\right)^{-4\sigma}\\ &\le 16^{\sigma}N2^{\tau(N)}3^{2\sigma+1}\sum_{k|N}\sum_{d\in\Omega}d^{1/2+\epsilon-2\sigma}(ku_k)^{-4\sigma}.\end{aligned}$$

Note that $\tau(n) = n^\epsilon$ as $n \to \infty$. Since, for a fixed positive integer v, there are at most $\tau(v^2-4)$ number of d's in Ω such that $v^2 - du^2 = 4$ for positive integers u, we have

$$\sum_{d\in\Omega} d^{1/2+\epsilon-2\sigma}(ku_k)^{-4\sigma} \le \sum_{d\in\Omega}(v_k^2-4)^{-\sigma} \le \sum_{v=3}^{\infty} \frac{\tau(v^2-4)}{(v^2-4)^\sigma} \ll \sum_{v=3}^{\infty}\frac{1}{(v^2-4)^{\sigma-\epsilon}} < \infty$$

for $\sigma > 1/2$. Hence, the series

$$\sum_{k|N} k^{1-2s} \sum_{d\in\Omega}\sum_{u\neq u_k} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k}(1+(\frac{d}{p}))\prod_{p|k}(1-(\frac{d}{p})p^{-1})\frac{h_d \ln\epsilon_d}{(du^2)^s}$$

represents an analytic function of s in the half-plane $\Re s > 1/2$. It follows from (2.4) that the function

$$\sum_{k|N} k^{1-2s} \sum_{d\in\Omega} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k}(1+(\frac{d}{p}))\prod_{p|k}(1-(\frac{d}{p})p^{-1})\frac{h_d \ln\epsilon_d}{(du_k^2)^s} \tag{2.7}$$

has analytic continuation to the half-plane $\Re s > 1/2$ except for having simple poles at $s = 1$ and $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$.

Next, let (v_0, u_0) be the smallest positive solution of Pell's equation

$$v^2 - du^2 = 4. \tag{2.8}$$

Let k be a divisor of N. If (v_k, ku_k) is a solution of (2.8) different from (v_0, u_0), then by Lemma 2.2 there exists an integer $n \ge 2$ such that

$$\frac{v_k + \sqrt{d}ku_k}{2} = \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^n.$$

Hence, we have

$$\begin{aligned}\sqrt{d}ku_k &= \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^n \left(1 - \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^{-2n}\right)\\ &> \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^n (1+2/\sqrt{d})^{-1}.\end{aligned} \tag{2.9}$$

By (2.9) and (2.3), we have

$$\begin{aligned}&|\sum_{k|N} k^{1-2s} \sum_{d\in\Omega,\, ku_k\neq u_0} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k}(1+(\frac{d}{p}))\prod_{p|k}(1-(\frac{d}{p})p^{-1})\frac{h_d \ln\epsilon_d}{(du_k^2)^s}|\\ &\le \sum_{k|N}\sqrt{kN}2^{\tau(N)} \sum_{d\in\Omega,\, ku_k\neq u_0} (1+2/\sqrt{d})^{2\sigma} h_d \ln\epsilon_d \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^{-2n\sigma}\\ &\le \sum_{k|N}\sqrt{kN}2^{\tau(N)}9^\sigma \sum_{d\in\Omega,\, ku_k\neq u_0} h_d \ln\epsilon_d \left(\frac{v_0+\sqrt{d}u_0}{2}\right)^{-4\sigma}\\ &\le 16^\sigma \tau(N) N 2^{\tau(N)} 9^\sigma \sum_{d\in\Omega} d^{1/2+\epsilon-2\sigma} u_0^{-4\sigma}.\end{aligned}$$

Since

$$\sum_{d\in\Omega} d^{1/2+\epsilon-2\sigma} u_0^{-4\sigma} \le \sum_{d\in\Omega} (v_0^2-4)^{-\sigma} \le \sum_{v=3}^{\infty} \frac{\tau(v^2-4)}{(v^2-4)^{\sigma}} \le \sum_{v=3}^{\infty} \frac{1}{(v^2-4)^{\sigma-\epsilon}} < \infty$$

for $\sigma > 1/2$, the series

$$\sum_{k|N} k^{1-2s} \sum_{d\in\Omega,\, ku_k\neq u_0} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1+(\frac{d}{p})) \prod_{p|k} (1-(\frac{d}{p})p^{-1}) \frac{h_d \ln \epsilon_d}{(du_k^2)^s}$$

represents an analytic function of s in the half-plane $\Re s > 1/2$. It follows from (2.7) that the function

$$\sum_{k|N} k \sum_{d\in\Omega,\, k|u_0} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1+(\frac{d}{p})) \prod_{p|k} (1-(\frac{d}{p})p^{-1}) \frac{h_d \ln \epsilon_d}{(du_0^2)^s} \tag{2.10}$$

has analytic continuation to the half-plane $\Re s > 1/2$ except for having simple poles at $s = 1$ and $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$, where the product on p^{2l} is over all distinct primes p with p^{2l} being the greatest even p-power factor of $(d, N/k)$. By Lemma 2.1 we can write (2.10) as

$$\sum_{k|N} \sum_{d\in\Omega,\, k|u_0} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} (1+(\frac{d}{p})) \frac{h_{dk^2} \ln \epsilon_{dk^2}}{(du_0^2)^s}.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 1. It is proved in [6] that the series

$$F(s) = \sum_{d\in\Omega} \frac{h_d \ln \epsilon_d}{d^s} \sum_{\substack{u>0 \\ v^2-du^2=4}} \frac{1}{u^{2s}},$$

represents an analytic function of s in the half-plane $\Re\, s > 1/2$ except for having a simple poles at $s = 1$. By (2.3), (3.4), (3.5), Lemma 3.5, Lemma 4.1, and Lemma 4.2 of [6], we have that

$$F(s) - h(-i/2) \tag{2.11}$$

is analytic in the half-plane $\Re\, s > 1/2$. Let (v_d, u_d) be the smallest positive solution of Pell's equation $v^2 - du^2 = 4$. If $u \neq u_d$, then

$$\frac{v+\sqrt{d}u}{2} = \left(\frac{v_d+\sqrt{d}u_d}{2}\right)^{\nu}$$

for some positive integer $\nu \geq 2$. Similarly as in (2.9), we can obtain that

$$\sqrt{d}u \geq \frac{1}{3}\epsilon_d^{\nu}.$$

It follows that

$$\left|\sum_{d\in\Omega}\frac{h_d \ln\epsilon_d}{d^s}\sum_{\substack{u\neq u_d\\ v^2-du^2=4}}\frac{1}{u^{2s}}\right| \leq 9\sum_{d\in\Omega}\frac{h_d\ln\epsilon_d}{\epsilon_d^{4\sigma}} \tag{2.12}$$
$$\leq 2^{4\sigma}9\sum_{d\in\Omega}d^{1/2+\epsilon-2\sigma}u_d^{-4\sigma} < \infty$$

for $\sigma = \Re s > 1/2$. Let

$$l(s) = \sum_{d\in\Omega}\frac{h_d\ln\epsilon_d}{(du_d^2)^s}.$$

By (2.11) and (2.12), we obtain that

$$l(s) - h(-i/2) \tag{2.13}$$

is analytic in the half-plane $\Re\, s > 1/2$.

Let $\bar{L}_N(s)$ be given as in Lemma 2.5. Then by (1.4), (4.4), (4.5), Theorem 4.3, Lemma 5.1, and Lemma 5.3 of [7], we have that

$$\bar{L}_N(s) - h(-i/2) \tag{2.14}$$

is an analytic function of s in the half-plane $\Re\, s > 1/2$ except for simple poles at $s = 1/2 - i\kappa_j$, $j = 1, 2, \cdots, S$. It follows from (2.13) and (2.14) that

$$L(s) = \bar{L}_N(s) - l(s)$$

represents an analytic function of s in the half-plane $\Re\, s > 1/2$ except for simple poles at $s_j = 1/2 - i\kappa_j$, $j = 1, 2, \cdots, S$. Moreover, we have

$$m_j = 2\,\mathrm{Res}_{s=s_j}L(s)$$

for $j = 1, 2, \cdots, S$.

This completes the proof of the theorem. $\square$

Proof of Corollary 2. By Theorem 1 the series

$$\sum_{k|N}k\sum_{d\in\Omega,\, k|u_d}\prod_{p^{2l}|(d,N/k)}p^l\prod_{p|N/k}\{1+\left(\frac{d}{p}\right)\}\prod_{p|k}\{1-\frac{1}{p}\left(\frac{d}{p}\right)\}\frac{h_d\ln\epsilon_d}{(du_d^2)^s} - \sum_{d\in\Omega}\frac{h_d\ln\epsilon_d}{(du_d^2)^s}$$

represents an analytic function of s in the half-plane $\Re s > 1/2$ except for having simple poles at $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$, where (v_d, u_d) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$ and the product on p^{2l} is over all distinct primes p with p^{2l} being the greatest even p-power factor of $(d, N/k)$. Moreover, we have

$$m_j = 2\,\mathrm{Res}_{s=s_j} L(s)$$

for $j = 1, 2, \cdots, S$.

Since N is square free, we have

$$\prod_{p|N/k} \{1 + \left(\frac{d}{p}\right)\} \prod_{p|k} \{1 - \frac{1}{p}\left(\frac{d}{p}\right)\} = \sum_{m|N} \frac{\mu((m,k))}{(m,k)} \left(\frac{d}{m}\right) \tag{2.15}$$

where $\mu(n)$ is the Möbius function and (m, k) denotes the greatest common divisor of m and k. By using the identity (2.15), we can write

$$\begin{aligned}
&\sum_{k|N} k \sum_{d\in\Omega,\, k|u_d} \prod_{p^{2l}|(d,N/k)} p^l \prod_{p|N/k} \{1 + \left(\frac{d}{p}\right)\} \prod_{p|k} \{1 - \frac{1}{p}\left(\frac{d}{p}\right)\} \frac{h_d \ln \epsilon_d}{(du_d^2)^s} - \sum_{d\in\Omega} \frac{h_d \ln \epsilon_d}{(du_d^2)^s} \\
&= \sum_{m|N, k|N} k \frac{\mu((m,k))}{(m,k)} \sum_{d\in\Omega,\, k|u_d} \left(\frac{d}{m}\right) \frac{h_d \ln \epsilon_d}{(du_d^2)^s} - \sum_{d\in\Omega} \frac{h_d \ln \epsilon_d}{(du_d^2)^s} \\
&= \sum_{\substack{m|N,\, k|N \\ (m,k)\neq(1,1)}} k \frac{\mu((m,k))}{(m,k)} \sum_{d\in\Omega,\, k|u_d} \left(\frac{d}{m}\right) \frac{h_d \ln \epsilon_d}{(du_d^2)^s}.
\end{aligned}$$

It follows that then the series

$$L_1(s) = \sum_{\substack{m|N,\, k|N \\ (m,k)\neq(1,1)}} k \frac{\mu((m,k))}{(m,k)} \sum_{d\in\Omega,\, k|u_d} \left(\frac{d}{m}\right) \frac{h_d \ln \epsilon_d}{(du_d^2)^s}$$

represents an analytic function of s in the half-plane $\Re s > 1/2$ except for having simple poles at those points $s_j = \frac{1}{2} - i\kappa_j$, $j = 1, 2, \cdots, S$, where (v_d, u_d) is the smallest positive solution of Pell's equation $v^2 - du^2 = 4$. Moreover, we have

$$m_j = 2\,\mathrm{Res}_{s=s_j} L_1(s)$$

for $j = 1, 2, \cdots, S$.

This completes the proof of the corollary. □

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